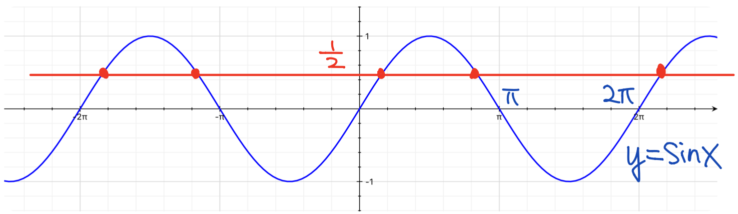
Math 1020 Week 5

Inverse functions

We want to define $\sin^{-1}x$.



Q What is $\sin^{-1}(\frac{1}{2})$?

$\frac{\pi}{6}$? $\frac{5\pi}{6}$? $\frac{13\pi}{6}$? $-\frac{7\pi}{6}$? $-\frac{11\pi}{6}$?

Which one?

Problem $\sin x$ is not one-to-one

Solution Need to restrict $\sin x$ to a smaller domain

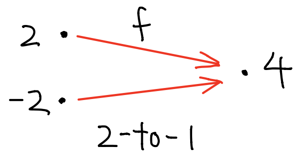
One-to-one function

$\boxed{\text{Defn } f(x) \text{ is called one-to-one if } f(x_1) \neq f(x_2) \text{ for } x_1 \neq x_2.}$

eg $f(x) = x^2$.

Note $2 \neq -2$ but $f(2) = 4 = f(-2)$

$\therefore$ f is not one-to-one



Equivalent defn

$\boxed{\text{Defn } f(x) \text{ is called one-to-one if } f(x_1) = f(x_2) \Rightarrow x_1 = x_2}$

eg $g(x) = 2x+3$

If $g(x_1) = g(x_2)$,

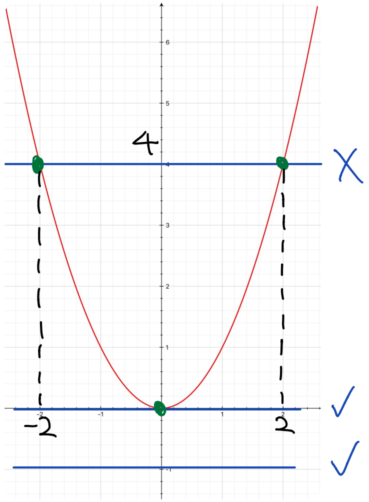
then $2x_1+3 = 2x_2+3 \Rightarrow 2x_1 = 2x_2$

$\Rightarrow x_1 = x_2$

$\therefore$ g is one-to-one

Horizontal line test

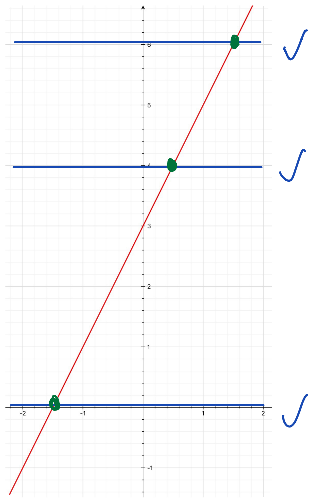
If every horizontal line has at most one intersection with the graph of $f(x)$ then f is one-to-one



Fails

$f(x) = x^2$

is not one-to-one



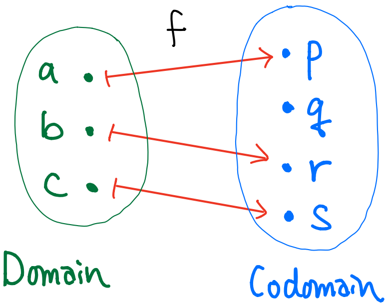
Passes

$g(x) = 2x+3$

is one-to-one

Prop If f is one-to-one, then its inverse f^{-1} can be defined

eg.



f is one-to-one, can define f^{-1} by

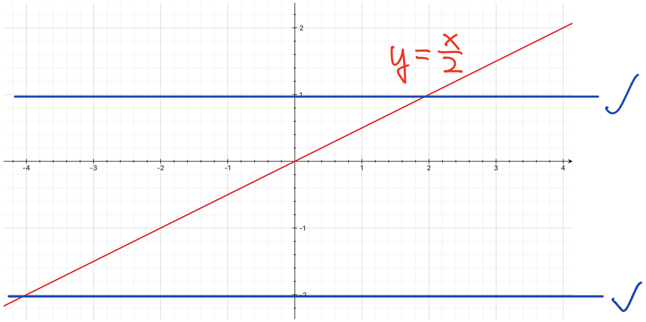
$f^{-1}(p) = a, \quad f^{-1}(r) = b, \quad f^{-1}(s) = c$

Prop $D_{f^{-1}} = R_f \quad R_{f^{-1}} = D_f$

$(f^{-1} \circ f)(x) = x$ for $x \in D_f$

$(f \circ f^{-1})(x) = x$ for $x \in D_{f^{-1}}$

eg $f(x) = \frac{x}{2}$



Passes horizontal line test

$\Rightarrow f$ is one-to-one

$\Rightarrow f^{-1}$ is defined

$$\begin{tikzcd} x \arrow[r, bend left, "f \;\; \times\frac{1}{2}"] & \frac{x}{2} \arrow[l, bend left, "f^{-1} \;\; \times 2"] \end{tikzcd}$$

Reverse Process :

$f^{-1}(x) = 2x$

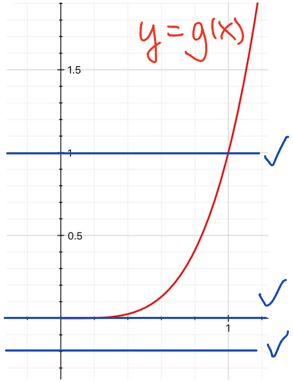
Also, $D_{f^{-1}} = R_f = (-\infty, \infty)$

$R_{f^{-1}} = D_f = (-\infty, \infty)$

eg x^4 is not one-to-one on $\mathbb{R}$

$\therefore$ To define inverse, need a smaller domain

Consider $g : [0, \infty) \longrightarrow \mathbb{R}$, $g(x) = x^4$



g is one-to-one

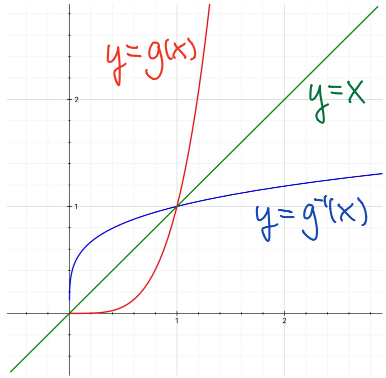
$g^{-1}(x) = \sqrt[4]{x}$,

$D_{g^{-1}} = R_g = [0, \infty)$

$R_{g^{-1}} = D_g = [0, \infty)$

Rmk

Graphs of $g(x)$ and $g^{-1}(x)$ are symmetric about $y = x$



Given one-to-one f, how to find f^{-1}?

1. Let $y = f(x)$
2. Express x in terms of y
3. If $x = g(y)$, then $f^{-1}(x) = g(x)$

eg Let $f(x) = \dfrac{3x+1}{x+2}$

Find $f^{-1}(x)$, its domain and range.

Sol Let $y = f(x) = \dfrac{3x+1}{x+2}$

$$y(x+2) = 3x+1$$

$$xy - 3x = 1 - 2y$$

$$x = \frac{1-2y}{y-3} = g(y)$$

$\therefore\ f^{-1}(x) = g(x) = \dfrac{1-2x}{x-3}$

$D_{f^{-1}} = \mathbb{R}\setminus\{3\}$

$R_{f^{-1}} = D_f = \mathbb{R}\setminus\{-2\}$

Rmk ① $R_f = D_{f^{-1}} = \mathbb{R}\setminus\{3\}$

② One can check answer:

$$f^{-1}(f(x)) = f^{-1}\left(\frac{3x+1}{x+2}\right)$$

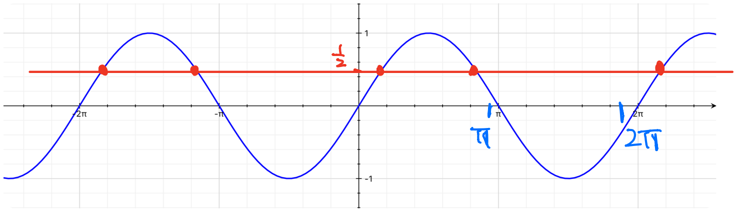
$$= \frac{1-2\left(\frac{3x+1}{x+2}\right)}{\frac{3x+1}{x+2} - 3}$$

$$= \frac{x+2-2(3x+1)}{3x+1-3(x+2)}$$

$$= \frac{-5x}{-5} = x \quad \checkmark$$

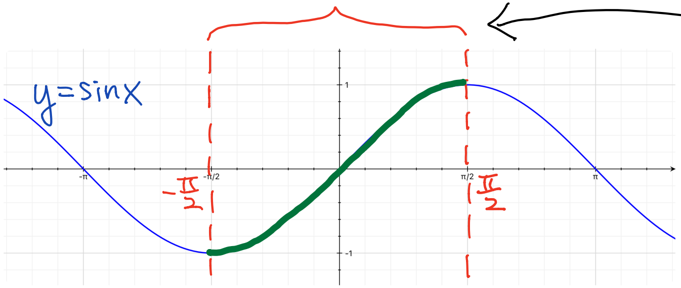
Back to trig. functions,

We want to define $\sin^{-1}$.



<u>Problem</u> $\sin x$ is not one-to-one on $\mathbb{R}$

<u>Solution</u> Restrict to a smaller domain



$\sin : [-\frac{\pi}{2}, \frac{\pi}{2}] \longrightarrow \mathbb{R}$ is <u>one-to-one</u>

its <u>range $[-1, 1]$, same as $\sin : \mathbb{R} \to \mathbb{R}$</u>

Define

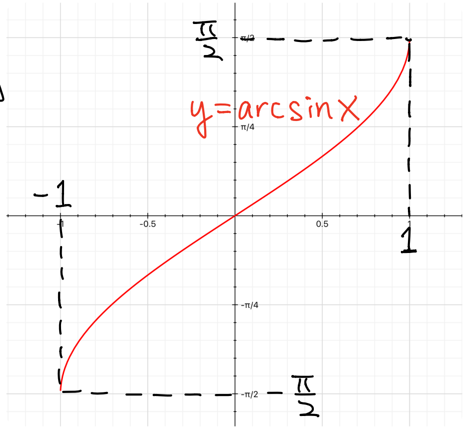
$$\boxed{\arcsin : [-1, 1] \longrightarrow \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]}$$

to be the inverse of $\sin x$ on $[-\frac{\pi}{2}, \frac{\pi}{2}]$

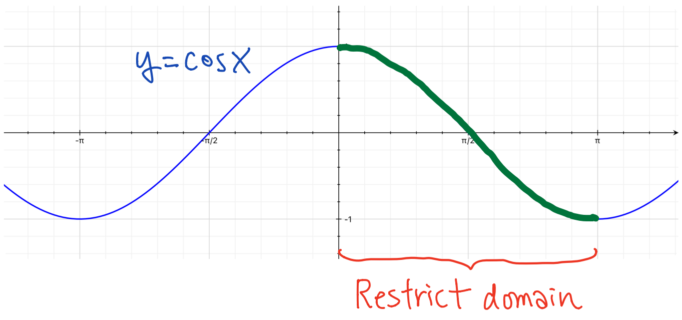
<u>eg</u> $\sin \frac{\pi}{4} = \frac{1}{\sqrt{2}} \longleftrightarrow \arcsin \frac{1}{\sqrt{2}} = \frac{\pi}{4}$

$\sin(-\frac{\pi}{6}) = -\frac{1}{2} \longleftrightarrow \arcsin\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$

Reflect across $y = x$

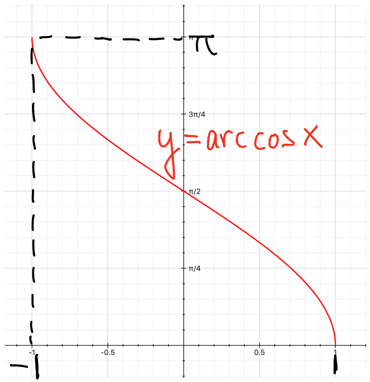


arccos

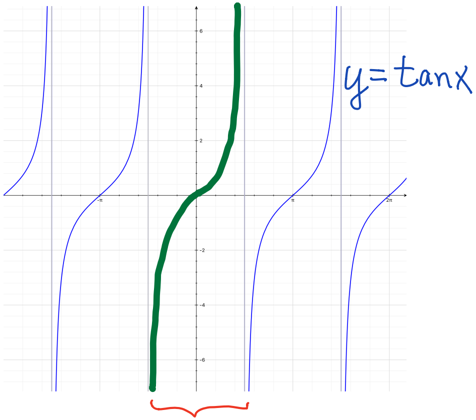


$\cos x$ is one-to-one on $[0,\pi]$ with range $[-1,1]$. Define its inverse:

$$\boxed{\arccos : [-1,1] \longrightarrow [0,\pi]}$$

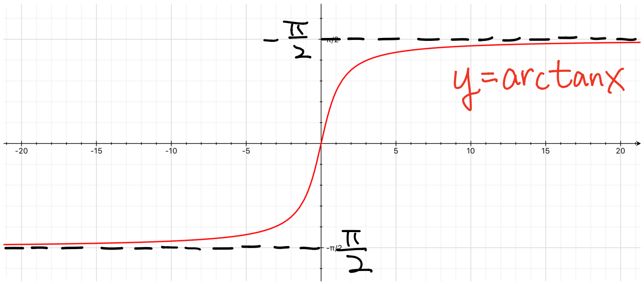


arctan



$\tan x$ is one-to-one on $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ with range $(-\infty,\infty)$. Define its inverse:

$$\boxed{\arctan : (-\infty,\infty) \longrightarrow \left(-\frac{\pi}{2},\frac{\pi}{2}\right)}$$



Summary

$\arcsin : [-1, 1] \longrightarrow [-\frac{\pi}{2}, \frac{\pi}{2}]$

$\arccos : [-1, 1] \longrightarrow [0, \pi]$

$\arctan : \mathbb{R} \longrightarrow (-\frac{\pi}{2}, \frac{\pi}{2})$

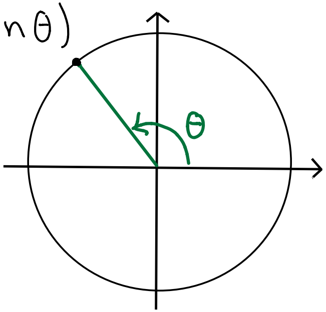
Other notation

$\sin^{-1} = \arcsin$

$\cos^{-1} = \arccos$

$\tan^{-1} = \arctan$

Recall: On unit circle



$$\boxed{\begin{array}{l} x = \cos\theta \\ y = \sin\theta \end{array}}$$

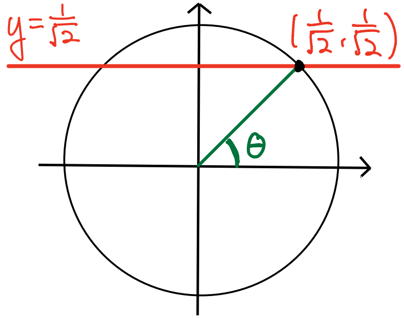
eg Find the following θ.

① $\theta = \arcsin(\frac{1}{\sqrt{2}})$

Sol

Range of $\arcsin = [-\frac{\pi}{2}, \frac{\pi}{2}]$

$$\left.\begin{array}{l} \therefore\ \theta \in [-\frac{\pi}{2}, \frac{\pi}{2}] \\ \text{Also, } \sin\theta = \frac{1}{\sqrt{2}} \end{array}\right\} \Rightarrow \theta = \frac{\pi}{4}$$



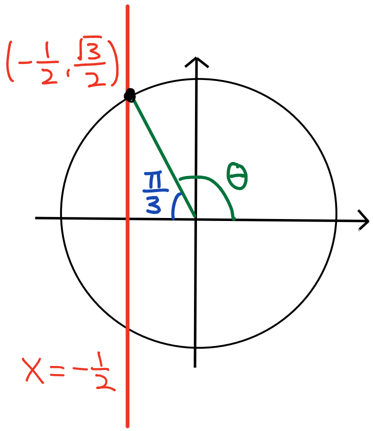
② $\theta = \arccos(-\frac{1}{2})$

Sol

$\theta \in R_{\arccos} = [0, \pi]$

$\cos\theta = -\frac{1}{2}$

$\Rightarrow \theta = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$

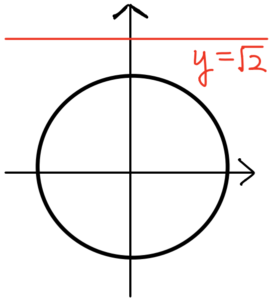


③ $\theta = \arcsin\sqrt{2}$

<u>Sol</u> $\sqrt{2} \notin [-1,1] = D_{\arcsin}$

$\Rightarrow$ $\arcsin\sqrt{2}$ is undefined

($\sin\theta \neq \sqrt{2}$ for any θ)



<u>eg</u> Evaluate $\sin\left(2\arccos\left(-\frac{2}{3}\right)\right)$

<u>Sol</u> Let $\theta = \arccos\left(-\frac{2}{3}\right)$

$$\sin\left(2\arccos\left(-\frac{2}{3}\right)\right) = \sin 2\theta$$

$$= 2\sin\theta\cos\theta$$

$$\cos\theta = -\frac{2}{3}$$

$$\sin^2\theta = 1-\cos^2\theta = 1-\left(-\frac{2}{3}\right)^2 = \frac{5}{9}$$

Note $\theta \in R_{\arccos} = [0,\pi]$

$\Rightarrow$ $\sin\theta \geqslant 0$

$\Rightarrow$ $\sin\theta = \frac{\sqrt{5}}{3}$

S	A
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$$\therefore \sin\left(2\arccos\left(-\frac{2}{3}\right)\right)$$

$$= 2\sin\theta\cos\theta$$

$$= 2\left(\frac{\sqrt{5}}{3}\right)\left(-\frac{2}{3}\right)$$

$$= -\frac{4\sqrt{5}}{9}$$